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Abstract. It is shown that $O(p, q)$ and $Sp(2, R)$ are complementary groups in the space of pseudo-oscillator H_{pq} eigenfunctions. The structure and irreducible representation of both invariancy algebra and generating-spectrum algebra are discussed in detail. It is proved that the transformation brackets between the basis diagonalising the compact generator and the basis diagonalising the non-compact generator in the case of discrete series of irreducible representations of the SU(1, 1) group coincide with the Clebsch–Gordan coefficients for the Kronecker product $D^{1/4+} \otimes D^{1/4-}$ of two irreducible representations of the SU(1, 1) group belonging to positive and negative discrete series respectively.

1. Introduction

A detailed analysis of the hyperspherical harmonic structure was made by Knyr *et al* (1975) on the basis of the complementary groups Sp(2, R) and $O(n)$ (Moshinsky and Quesne 1971). In particular it was shown that n-dimensional hyperspherical harmonics may be constructed from the hyperspherical harmonics for subspaces with fewer dimensions by using vector coupling of the non-compact moments for the Sp(2, R) group. The transformation brackets between the bases corresponding to different types of reduction of the $O(n)$ group to its subgroups (the so called 'tree' coefficients) coincide with the $3mj$-symbols for the Sp(2, R) (or SU(1, 1)) group. These conclusions were obtained by analysing the n-dimensional harmonic oscillator system. In this case we dealt only with the positive discrete series of the unitary irreducible representations (UIR) of the SU(1, 1) group, including the ray UIR with angular moments j which are multiples of $\frac{1}{4}$.

The concept of complementary groups (Moshinsky and Quesne 1971) may be used for the analysis of the hyperspherical harmonic structure of the non-compact $O(p, q)$ group. In this case, it is necessary to investigate the properties of the pseudo-oscillator system with the Hamiltonian

$$H_{pq} = \sum_{i=1}^{p} H_i - \sum_{k=p+1}^{p+q} H_k, \tag{1.1}$$

where

$$H_s = \tfrac{1}{2}(p_s^2 + x_s^2), \qquad s = 1, 2, \ldots, p+q$$

is the usual Hamiltonian of the linear harmonic oscillator.

One can use such a Hamiltonian in the relativistic quark model (Cocho and Flores 1971, Feynman *et al* 1971) for calculations of relativised form factors for elastic and inelastic electron scattering by nuclei (Cocho and Mondragon 1969, Cocho and Flores

1970a, b). In addition, the $O(p, q)$ group (which is the symmetry group of the Hamiltonian (1.1)) is connected very closely with general relativistic problems. In particular, it is interesting to perform the mapping of the continuum spectrum of the Coulomb and Coulomb–Dirac problems onto the pseudo-oscillator in the spirit of the calculations performed by Moshinsky (1971) and Basu (1971).

In § 2 it will be shown that $Sp(2, R)$ and $O(p, q)$ are complementary groups. We then consider the simplest example: the two-dimensional pseudo-oscillator. The symmetry group for this system will be discussed in § 3, and the UIR basis of the $SU(1, 1)$ group diagonalising the non-compact generator will be constructed.

The dynamical group of the Hamiltonian H_{11} is described in § 4. It will be shown that the transformation brackets between the usual Cartesian wavefunctions of a two-dimensional harmonic oscillator and the basic functions mentioned above must coincide with the Clebsch–Gordan coefficients for the Kronecker product $D^{1/4+} \otimes D^{1/4-}$ of two UIR of the $SU(1, 1)$ group belonging to the positive and negative discrete series respectively. The application of this approach to the arbitrary pseudo-oscillator system will be described in future publications.

2. Complementary groups Sp(2, R) and O(p, q)

Let us introduce the usual creation and annihilation operators

$$a_i^+ = \frac{1}{\sqrt{2}}\left(x_i - \frac{\partial}{\partial x_i}\right), \qquad a_i = \frac{1}{\sqrt{2}}\left(x_i + \frac{\partial}{\partial x_i}\right), \qquad i = 1, 2, \ldots, p,$$

$$a_\alpha^+ = \frac{1}{\sqrt{2}}\left(x_\alpha - \frac{\partial}{\partial x_\alpha}\right), \qquad a_\alpha = \frac{1}{\sqrt{2}}\left(x_\alpha + \frac{\partial}{\partial x_\alpha}\right), \qquad \alpha = p+1, p+2, \ldots, p+q \tag{2.1}$$

with the standard commutation rules

$$[a_s, a_{s'}^+] = \delta_{ss'}, \qquad [a_s, a_{s'}] = [a_s^+, a_{s'}^+] = 0, \qquad s, s' = 1, 2, \ldots, p+q.$$

The operator of infinitesimal rotation in the Euclidean plane (x_i, x_j) (or (x_α, x_β)) may be represented in the form

$$\begin{aligned} L_{ij} &= x_i(\partial/\partial x_j) - x_j(\partial/\partial x_i) = a_i^+ a_j - a_j^+ a_i, \\ L_{\alpha\beta} &= a_\alpha^+ a_\beta - a_\beta^+ a_\alpha. \end{aligned} \tag{2.2}$$

The operator of infinitesimal rotation in the non-Euclidean plane (x_i, x_α) may be written as

$$L_{i\alpha} = x_i(\partial/\partial x_\alpha) + x_\alpha(\partial/\partial x_i) = a_i^+ a_\alpha^+ - a_i a_\alpha. \tag{2.3}$$

The angular part of the Laplacian Λ in equation (1.1) (i.e. the Casimir operator for the $O(p, q)$ group) is of the form

$$\Lambda = \tfrac{1}{4}(L_{ij}L_{ji} + L_{i\alpha}L_{\alpha i} + L_{\alpha j}L_{j\alpha} + L_{\alpha\beta}L_{\beta\alpha}). \tag{2.4}$$

Here and in the remainder of this section the repeated indices mean the summation from 1 to p for italic indices and from $p+1$ to $p+q$ for Greek indices. The eigenvalues λ of this operator will be given by the expression $\lambda = k(k+n-2)$, $(n = p+q)$.

The dynamical group of the Hamiltonian (1.1) is the Sp(2, R) group with generators

$$I_+ = \tfrac{1}{2}\mathrm{i}(a_i^+ a_i^+ - a_\alpha a_\alpha), \qquad I_- = \tfrac{1}{2}\mathrm{i}(a_i a_i - a_\alpha^+ a_\alpha^+),$$
$$I_0 = \tfrac{1}{2}[a_i^+ a_i - a_\alpha^+ a_\alpha + \tfrac{1}{2}(p-q)] = \tfrac{1}{2}H_{pq}. \tag{2.5}$$

These generators have the properties

$$(I_\pm)^+ = -I_\mp, \qquad I_0^+ = I_0, \qquad [I_+, I_-] = 2I_0, \qquad [I_0, I_\pm] = \pm I_\pm. \tag{2.6}$$

The Casmir operator Q for the Sp(2, R) group is determined by the expression

$$Q = I_- I_+ + I_0^2 + I_0. \tag{2.7}$$

Its eigenvalues will be designated as $j(j+1)$, and in particular we have

$$j = -\tfrac{1}{2} + \mathrm{i}\sigma, \qquad \sigma \in (-\infty, +\infty) \tag{2.8}$$

for UIR of the principal continuous series realised in the pseudo-oscillator case.

Comparing the Casimir operator of the O(p, q) and S$q(2, R)$ groups (equations (2.4) and (2.7)) it is easy to find that

$$Q = \tfrac{1}{4}\Lambda + \tfrac{1}{16}n^2 - \tfrac{1}{4}n. \tag{2.9}$$

Therefore fixing the UIR D^j of the S$_p$(2, R) group simultaneously determines the UIR D^k of the O(p, q) group. Hence these two groups are complementary groups, and

$$j = \tfrac{1}{2}k + \tfrac{1}{4}n - 1. \tag{2.10}$$

It is clear from this equation that in the pseudo-oscillator system the UIR D^k of the O(p, q) group may be realised with

$$k = 2\mathrm{i}\sigma - \tfrac{1}{2}(n-2). \tag{2.11}$$

Let us now turn to the simplest two-dimensional pseudo-oscillator.

3. Symmetry group of the two-dimensional pseudo-oscillator

The symmetry group of the Hamiltonian $H_{11} = H_1 - H_2$ is the SU(1, 1)$_s$ group with generators

$$J_1 = \tfrac{1}{2}\mathrm{i}(a_1^+ a_2^+ + a_1 a_2), \qquad J_2 = \tfrac{1}{2}(a_1^+ a_2^+ - a_1 a_2), \qquad J_0 = \tfrac{1}{2}(a_1^+ a_1 + a_2^+ a_2 + 1). \tag{3.1}$$

We shall label the symmetry and dynamical SU(1, 1) groups by indices 's' and 'd', respectively.

The generators (3.1) satisfy the relations

$$(J_{1,2})^+ = -J_{1,2}, \qquad J_0^+ = J_0, \qquad [J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k, \qquad [J_i, H_{11}] = 0. \tag{3.2}$$

We shall also need the generators

$$J_+ = \mathrm{i}a_1^+ a_2^+ = (-J_-)^+, \qquad J_- = \mathrm{i}a_1 a_2 = (-J_+)^+,$$
$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0. \tag{3.3}$$

The eigenvalue problem

$$H_{11}\psi(x_1, x_2) = n\psi(x_1, x_2) \tag{3.4}$$

can be separated into Cartesian variables x_1, x_2 and we have

$$\psi_n(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2) \tag{3.5}$$

as the eigenfunctions of H_{11}. Here $n = n_1 - n_2 = 0, \pm1, \pm2, \ldots, n_1, n_2 = 0, 1, 2, \ldots$, and $\psi_{n_i}(x_i)$ are the usual wavefunctions for the linear harmonic oscillator.

The following properties can be easily proved:

$$\begin{aligned} &G|n_1n_2\rangle = \tfrac{1}{4}(n^2-1)|n_1n_2\rangle, && G = J_-J_+ + J_0^2 + J_0, \\ &J_0|n_1n_2\rangle = \tfrac{1}{2}(N+1)|n_1n_2\rangle, && N = n_1 + n_2, \end{aligned} \tag{3.6}$$

i.e. the eigenfunctions $|n_1n_2\rangle$ with fixed $n = n_1 - n_2$ and all possible $N = n_1 + n_2 = n$, $n+2$, $n+4, \ldots$ belong to the UIR D^{J+} of the $SU(1, 1)_s$ group with $J = \frac{1}{2}(|n| - 1)$. It should be noted that each positive discrete series UIR is presented in the spectrum of H_{11} twice (except for $n = 0$, $J = -\frac{1}{2}$), because the eigenvalues n and $-n$ correspond to the same eigenvalues of the Casimir operator G. By direct calculation we obtain

$$\begin{aligned} &J_\pm|n_1n_2\rangle \equiv J_\pm|JM\rangle = [(J\mp M)(J\pm M+1)]^{\frac{1}{2}}|JM\pm1\rangle, \\ &J = \tfrac{1}{2}(|n_1-n_2|-1), \qquad M = \tfrac{1}{2}(n_1+n_2+1). \end{aligned} \tag{3.7}$$

Equation (3.4) can also be separated into hyperbolic coordinates

$$\begin{aligned} &\left.\begin{aligned} x_1 &= r\cosh\varphi \\ x_2 &= r\sinh\varphi \end{aligned}\right\} && \begin{aligned} &\text{in sector I} \\ &x_1 > |x_2| \end{aligned} \\ &\left.\begin{aligned} x_1 &= -r\cosh\varphi \\ x_2 &= -r\sinh\varphi \end{aligned}\right\} && \begin{aligned} &\text{in sector II} \\ &|x_2| < -x_1 \end{aligned} \\ &\left.\begin{aligned} x_1 &= r\sinh\varphi \\ x_2 &= r\cosh\varphi \end{aligned}\right\} && \begin{aligned} &\text{in sector III} \\ &x_2 > |x_1| \end{aligned} \\ &\left.\begin{aligned} x_1 &= -r\sinh\varphi \\ x_2 &= -r\cosh\varphi \end{aligned}\right\} && \begin{aligned} &\text{in sector IV} \\ &-x_2 > |x_1|. \end{aligned} \end{aligned} \tag{3.8}$$

Here $0 \leqslant r < \infty$, $-\infty < \varphi < \infty$.

The Hamiltonian H_{11} is of the form

$$H_{11} = \tfrac{1}{2}\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + r^2\right] \tag{3.9}$$

in sector I. Since x_1 and x_2 change their signature in sector II, the Hamiltonian conserves the same form (3.9) in this sector. In sector III there is a permutation of x_1 and x_2; therefore H_{11} has the form (3.9) with the opposite signature. In sector IV H_{11} has the same form as in sector III. Therefore we can seek the solution of (3.4) only in sectors I and III, because the solutions for sectors II and IV can be found by the reflection of axes x_1 or x_2.

As a result we have the following equations:

$$\frac{1}{2}\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + r^2\right]\phi_{1,2} = \pm E\phi_{1,2}. \tag{3.10}$$

Here $\phi_1(x_1, x_2)$ is the wavefunction in sector I ($x_1^2 - x_2^2 > 0$), and $\phi_2(x_1, x_2)$ is the wavefunction in sector III ($x_1^2 - x_2^2 < 0$). Since our purpose is to compare the solutions of equation (3.4) in both Cartesian and hyperbolic coordinates, we shall seek the solutions of equation (3.10) which correspond to the same eigenvalues $E = n = 0, \pm 1, \pm 2, \dots$ as in the case of Cartesian coordinates. Besides, we shall assume that

$$\phi_i(r, \varphi) \to 0, \qquad i = 1, 2 \tag{3.11}$$

for $r \to \infty$. Such behaviour is connected with the fact that the functions $|n_1 n_2\rangle$ have asymptotic properties $\sim r^N \exp[(-r^2 \operatorname{ch} 2\varphi)/2]$ in each sector of the plane (x_1, x_2). Let us separate the angular and radial variables:

$$\phi_i(r, \varphi) = R_i(r)\psi_i(\varphi). \tag{3.12}$$

This corresponds to the diagonalisation of the non-compact generator $J_2 = \frac{1}{2}L_{12} = -\frac{1}{2}(\partial/\partial\varphi)$. Since J_2 is anti Hermitian, it has the imaginary eigenvalues

$$J_2[\psi_\sigma(\varphi)]_i = \mathrm{i}\sigma[\psi_\sigma(\varphi)]_i.$$

Therefore we obtain

$$[\psi_\sigma(\varphi)]_{1,2} = (1/\sqrt{\pi})\, \mathrm{e}^{-2\mathrm{i}\sigma\varphi}, \qquad -\infty < \sigma < \infty. \tag{3.13}$$

Substituting (3.12), (3.13) into equation (3.10) we can find the following equation for the radial wavefunctions,

$$\frac{1}{2}\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + r^2 - \frac{4\sigma^2}{r^2} \mp 4n\right)[R_{n\sigma}(r)]_i = 0, \tag{3.14}$$

where the upper signature corresponds to sector I and the lower one corresponds to sector III.

After the transformation $[R_{n\sigma}(r)]_i = r^{-1}[\chi_{n\sigma}(\xi)]_i$, $\xi = r^2$, equation (3.14) reduces to the usual Whittaker equation (Bateman 1953)

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\chi_i(\xi) + \left(-\frac{1}{4} + \frac{(\pm n)}{\xi} + \frac{\frac{1}{4} - (\mathrm{i}\sigma)^2}{\xi^2}\right)\chi_i(\xi) = 0.$$

Taking into account the boundary conditions (3.11), we can write the solution of our problem in the form

$$\phi_{n\sigma}(r, \varphi) = \begin{pmatrix} C_{1n} W_{n/2,\mathrm{i}\sigma}(r^2)/r \\ C_{2n} W_{-n/2,\mathrm{i}\sigma}(r^2)/r \end{pmatrix} (1/\sqrt{\pi})\, \mathrm{e}^{-2\mathrm{i}\sigma\varphi}, \tag{3.15a}$$

where

$$G_{1n} = \{\Gamma[(1+n)/2 - \mathrm{i}\sigma]\}^{-1}, \qquad C_{2n} = \{\Gamma[(1-n)/2 - \mathrm{i}\sigma]\}^{-1}. \tag{3.15b}$$

The wavefunctions (3.15*a*) are characterised by the following properties:

(1) $P_{12}\phi_{n\sigma}(x_1, x_2) = \phi_{n\sigma}(x_2, x_1) = \phi_{-n\sigma}(x_1, x_2)$;

(2) These functions form the standard basis of the UIR of the SU(1, 1) group, as will be shown later in § 3;

(3) Finally, according to Montgomery and O'Raifertaigh (1974), the wavefunctions (3.15*a*) are orthonormalised:

$$\int_0^\infty \frac{\mathrm{d}r}{r} \frac{W_{n/2,\mathrm{i}\sigma}(r^2) W_{n'/2,\mathrm{i}\sigma}(r^2)}{\Gamma[(1+n)/2 + \mathrm{i}\sigma]\Gamma[(1+n')/2 - \mathrm{i}\sigma]} + \int_0^\infty \frac{\mathrm{d}r}{r} \frac{W_{-n/2,\mathrm{i}\sigma}(r^2) W_{-n'/2,\mathrm{i}\sigma}(r^2)}{\Gamma[(1-n)/2 + \mathrm{i}\sigma]\Gamma[(1-n')/2 - \mathrm{i}\sigma]} = \delta_{nn'}, \tag{3.16}$$

$$\langle n\sigma | n'\sigma'\rangle = \delta_{nn'}\delta(\sigma - \sigma'). \tag{3.17}$$

The functions $|n\sigma\rangle$ with fixed n are the basis vectors of the UIR $D^{J+}(J=\frac{1}{2}(|n|-1))$. This basis is continuous, because the non-compact generator J_2 is diagonalised. It corresponds to the reduction of the $\mathrm{SU}(1,1)_s$ group to the $\mathrm{O}(1,1)$ subgroup. Equation (3.14) contains the attractive potential r^{-2}. In such cases, it is necessary to expand the Hamiltonian to the self-conjugated form (Case 1950). Obviously we can solve this problem by choosing the equidistant spectrum of H_{11} and the boundary conditions (3.11).

We are interested in the transformation brackets $\langle n_1n_2|n\sigma\rangle$ between two types of bases for the UIR D^{J+} of the $\mathrm{SU}(1,1)_s$ group. A similar problem was analysed by Montgomery and O'Raifertaigh (1974), but in their work the continual basis was obtained by the diagonalisation of the operator $J_1+\mathrm{i}J_0$. Therefore the calculation of $\langle n_1n_2|n\sigma\rangle$ is a new problem. However, it is reasonable to discuss first the dynamical group of the Hamiltonian H_{11}, which allows us to look at the transformation brackets from another point of view.

4. Dynamical group of the two-dimensional pseudo-oscillator

The dynamical group $\mathrm{SU}(1,1)_d$ of the Hamiltonian H_{11} is determined by the generators

$$I_+=\tfrac{1}{2}\mathrm{i}(a_1^+a_1^+-a_2a_2), \qquad I_-=\tfrac{1}{2}\mathrm{i}(a_1a_1-a_2^+a_2^+),$$
$$I_0=\tfrac{1}{2}(a_1^+a_1-a_2^+a_2)=\tfrac{1}{2}H_{11}. \tag{4.1}$$

In accordance with (2.9) we have

$$Q|n\sigma\rangle=-(\sigma^2+\tfrac{1}{4})|n\sigma\rangle, \qquad Q=J_2-\tfrac{1}{4}.$$

Hence the functions $|n\sigma\rangle$ with fixed σ and all possible $n=0,\pm1,\pm2,\ldots$ form a basis of the UIR $D^j(j=-\frac{1}{2}+\mathrm{i}\sigma)$ of the $\mathrm{SU}(1,1)_d$ group. It should be noted that the even values of n correspond to the first principal continuous series of UIR, and the odd n belong to the second principal series. Each UIR D^j is contained in the spectrum of H_{11} twice, because the reverse of the signature of σ does not change the eigenvalue of the Casimir operator.

The generators (4.1) can be rewritten in hyperbolic coordinates as (sector I)

$$I_+=\tfrac{1}{2}\mathrm{i}(r^2-1-r(\partial/\partial r)-H_{11}), \qquad I_-=\tfrac{1}{2}\mathrm{i}(r^2+1+r(\partial/\partial r)-H_{11}) \tag{4.2a}$$

or

$$I_1=\tfrac{1}{2}\mathrm{i}(r^2-H_{11}), \qquad I_2=-\tfrac{1}{2}(r(\partial/\partial r)+1), \qquad I_1+\mathrm{i}I_0=\tfrac{1}{2}\mathrm{i}r^2. \tag{4.2b}$$

In sector III, I_+ and I_- change their roles.

It was shown by Mukunda and Radhakrishnan (1972, 1974) that the UIR of the $\mathrm{SU}(1,1)_d$ group which belongs to the principal series can be realised by using the eigenfunctions of the two-dimensional pseudo-oscillator. The generator I_2 was diagonalised, and the basis corresponding to the reduction $\mathrm{SU}(1,1)_d\supset\mathrm{O}(1,1)$ was obtained by Mukunda and Radhakrishnan (1972, 1974). We are interested in the $\mathrm{SU}(1,1)_d\supset\mathrm{O}(2)$ reduction that is realised by the basis functions (3.15a). In fact by using the recurrent relations (Bateman 1953, Gradshteyn and Ryzhik 1963)†

$$z(\partial/\partial z)W_{\lambda,\mu}(z)=(\lambda-\tfrac{1}{2}z)W_{\lambda,\mu}(z)-[\mu^2-(\lambda-\tfrac{1}{2})^2]W_{\lambda-1,\mu}(z),$$
$$(z-2\lambda)W_{\lambda,\mu}(z)=[(\lambda-\tfrac{1}{2})^2-\mu^2]W_{\lambda-1,\mu}(z)+W_{\lambda+1,\mu}(z), \tag{4.3}$$

† The second recurrent relation for the Whittaker functions is absent in these textbooks, but it may easily be proved by using the recurrent relations for the confluent hypergeometrical functions $\Phi(a;c;z)$.

we obtain the expressions

$$I_{\pm}|n\sigma\rangle = [(j \mp m)(j \pm m+1)]^{1/2}|n \pm 1\sigma\rangle(-)^{\epsilon},$$
$$j = -\tfrac{1}{2}+\mathrm{i}\sigma, \qquad m = \tfrac{1}{2}n. \tag{4.4}$$

It means that the vectors $|n\sigma\rangle$ form the standard basis of the UIR of the $\mathrm{SU}(1,1)_{\mathrm{d}}$ group belonging to the principal series (except for the inessential phase factor $(-)^{\epsilon}$).

Now let us ask ourselves what is the basis $|n_1n_2\rangle$ in relation to the $\mathrm{SU}(1,1)_{\mathrm{d}}$ group. It should be noted that the operators $\frac{1}{2}\mathrm{i}a_1^+a_1^+$, $\frac{1}{2}\mathrm{i}a_1a_1$, $\frac{1}{2}(a_1^+a_1+\frac{1}{2})$ are the generators of the UIR $D^{1/4+}$ of the $\mathrm{SU}(1,1)_{\mathrm{d}}$ group (Knyr *et al* 1975). The wavefunctions $\psi_{n_1}(x_1)$ of the linear harmonic oscillator are the basis vectors of this UIR. On the other hand, the operators $-\frac{1}{2}\mathrm{i}a_2^+a_2^+$, $-\frac{1}{2}\mathrm{i}a_2a_2$, $-\frac{1}{2}(a_2^+a_2+\frac{1}{2})$ can be considered as generators of the UIR $D^{1/4-}$ of the $\mathrm{SU}(1,1)_{\mathrm{d}}$ group. The functions $\psi_{n_2}(x_2)$ are the basis of this UIR belonging to negative discrete series. Hence a set of functions $|n_1n_2\rangle$ is a basis for the Kronecker product $D^{1/4+}\otimes D^{1/4-}$ of two UIR of the $\mathrm{SU}(1,1)_{\mathrm{d}}$ group. This product is reducible and can be expanded in terms of the UIR of the principal series (Mukunda and Radhakrishnan 1974):

$$D^{1/4+}\otimes D^{1/4-} = \int \mathrm{d}\sigma\, D^{-1/2+\mathrm{i}\sigma}. \tag{4.5}$$

Therefore the transformation brackets $\langle n_1n_2|n\sigma\rangle$ represent the Clebsch–Gordan coefficients for the $\mathrm{SU}(1,1)_{\mathrm{d}}$ group:

$$\langle n_1n_2|n\sigma\rangle \equiv [-\tfrac{1}{4}^+, \tfrac{1}{2}n_1;\ -\tfrac{1}{4}^-, -\tfrac{1}{2}n_2| -\tfrac{1}{2}+\mathrm{i}\sigma, \tfrac{1}{2}(n_1-n_2)]. \tag{4.6}$$

They may be calculated by using the general algebraic formulae given by Holman and Biedenharn (1966). The following properties characterise these coefficients:

$$\sum_{n_1n_2} \langle n_1n_2|n\sigma\rangle\langle n\sigma'|n_1n_2\rangle = \delta(\sigma-\sigma'),$$
$$\int \mathrm{d}\sigma \sum_{n} \langle n_1n_2|n\sigma\rangle\langle n\sigma|n_1'n_2'\rangle = \delta_{n_1n_1'}\delta_{n_2n_2'}, \tag{4.7}$$

$$\frac{1}{\sqrt{\pi}}\int \mathrm{d}\varphi\, \mathrm{e}^{2\mathrm{i}\sigma\varphi}|n_1n_2\rangle = \langle n_1n_2|n\sigma\rangle R_{n\sigma}(r). \tag{4.8}$$

It is easy to prove directly the validity of the last formula at $n_1 = n_2 = n = 0$:

$$\frac{1}{\sqrt{\pi}}\int \mathrm{d}\varphi\, \exp(2\mathrm{i}\sigma\varphi)\exp(-\tfrac{1}{2}r^2\cosh\varphi) = \frac{1}{\pi}K_{\mathrm{i}\sigma}\left(\frac{r^2}{2}\right) = \frac{1}{\sqrt{\pi}}W_{0,\mathrm{i}\sigma}(r^2)\frac{1}{r}.$$

In the general case, a set of new relations between the special functions K_s, $W_{\lambda,\mu}$, etc may be obtained on the basis of equation (4.8). It can also be considered as the separation of the radial part of the two-dimensional pseudo-oscillator wavefunction. In this way it is possible to obtain the analogue of integral transformations (Barut and Girardello 1971, Bargmann 1961, 1967, Kramer *et al* 1977) for the hyperbolic coordinate system. In conclusion, it should be noted that the self-reproducing kernel for $|n\sigma\rangle$ functions (i.e. for the Whittaker functions) is the hyperdifferential operator $\exp[\mathrm{i}z(H_1-H_2)]$ connected with the two-dimensional Fourier transformation (Wolf 1976).

The approach developed in this paper can be easily generalised to an arbitrary pseudo-oscillator H_{pq}, and it will be useful for the analysis of the $\mathrm{O}(p,q)$ spherical harmonics. The construction of these harmonics may be fulfilled by means of the 'vector coupling' procedure for the non-compact angular momenta belonging to discrete or principal series of UIR of the $SU(1, 1)_d$ group.

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